Question 1 **(10 marks)**

(a) Given $z_1=1, z_2=1+i$ and $z_3=i$, show that $|z_2|=|z_3-z_1|$ and that

$$\arg\left(\frac{z_3-z_1}{z_2}\right)=\frac{\pi}{2}.$$

(b) Sketch the region, where $\arg(z-1)\le\frac{\pi}{3}$ and $|z-1|\ge 2$, hold simultaneously.

(c) Given that $\sin(\theta+\alpha)=2\sin(\theta-\alpha)$, prove $\tan\theta=3\tan\alpha$.

Question 2 **(10 marks)**

(a) If $\frac{iz-1}{z-i}$ is real, find the value of $|z|$, where $z=a+ib$.

(b) Find the value of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{12}$, give your answer in the form $a+ib$.

(c) Expand $\left(z+\frac{1}{z}\right)^3$, for $z=\cos\theta+i\sin\theta$, and hence show that

$$\cos 3\theta=4\cos^3\theta-3\cos\theta$$

Question 3 **(10 marks)**

(a) (i) Prove that $\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$, given that $t=\tan\frac{\theta}{2}$.

(ii) Hence prove that $\sin 2\theta=\frac{4t(1-t^2)}{(1+t^2)^2}$ and thus find the roots of the equation $t^4+8t^3+2t^2-8t+1=0$.

Question 1

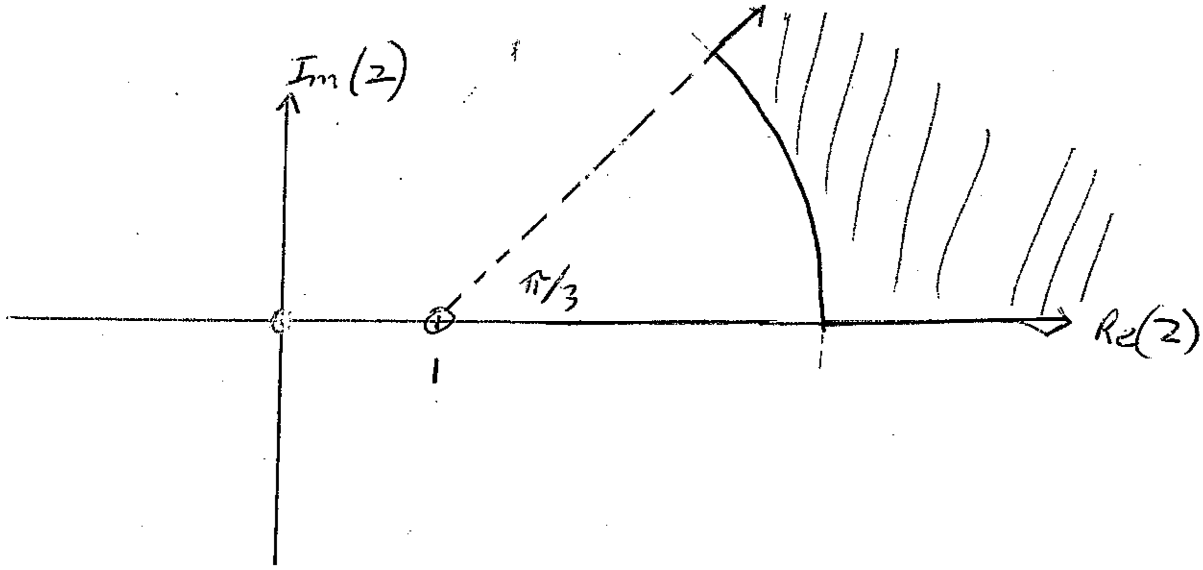
a) $|z_2| = \sqrt{1^2+1^2} = \sqrt{2}$

$|z_3 - z_1| = |i-1| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\arg\left(\frac{z_3-z_1}{z_2}\right) = \arg\left(\frac{(-1+i)}{1+i}\right) = \arg(-1+i) - \arg(1+i)$

$= \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$

b)



Question 2

a) $\frac{i(a+ib)-1}{a+ib-i} = \frac{-b-1+ia}{a+i(b-1)} \times \frac{a-i(b-1)}{a-i(b-1)}$

$= \frac{-ba+ib^2-ib-a+ib-i+ia^2+ab-a}{a^2+(b-1)^2}$

$= \frac{-ba-a+ab-a+i(b^2-b+b+a^2-1)}{a^2+(b-1)^2}$

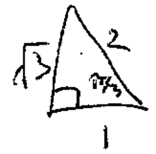
$= \frac{-2a+i(a^2+b^2-1)}{a^2+(b-1)^2}$

if $\frac{iz-1}{z-i}$ is real $\quad a^2+b^2-1=0 \therefore a^2+b^2=1$

$|z| = \sqrt{a^2+b} = \sqrt{1} = 1$

Question 2

b) $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\ \text{cis}\left(\tan^{-1} \frac{\sqrt{3}/2}{1/2}\right)$



$$= \sqrt{1}\ \text{cis}\ \frac{\pi}{3}$$

$$= \text{cis}\ \frac{\pi}{3}$$

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{12} = \left(\text{cis}\ \frac{\pi}{3}\right)^{12}$$

$$= \text{cis}\ \frac{12\pi}{3} = \text{cis}\ 4\pi = \text{cis}\ 0$$

$$= \cos 0 + i\sin 0$$

$$= 1$$

c) $\left(z + \frac{1}{z}\right)^3 = \left(\text{cis}\,\theta + \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}\right)^3$

$$= \left(\text{cis}\,\theta + \frac{\cos\theta - i\sin\theta}{1}\right)^3$$

$$= \left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right)^3$$

1
1 1
1 2 1
1 3 3 1

$$= (2\cos\theta)^3$$

$$= 8\cos^3\theta$$

$$\left(z + \frac{1}{z}\right)^3 = z^3 + 3z^2\left(\frac{1}{z}\right) + 3(z)\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3$$

$$= z^3 + 3z + 3z^{-1} + z^{-3}$$

$$= (\cos\theta + i\sin\theta)^3 + 3(\cos\theta + i\sin\theta) + 3(\cos\theta + i\sin\theta)^{-1} + (\cos\theta + i\sin\theta)^{-3}$$

$$= \cos 3\theta + i\sin 3\theta + 3\cos\theta + i3\sin\theta + 3\cos(-\theta) + i3\sin(-\theta)$$
$$+ \cos(-3\theta) + i\sin(-3\theta)$$

$$= \cos 3\theta + i\sin 3\theta + 3\cos\theta + i3\sin\theta + 3\cos\theta - i3\sin\theta$$
$$+ \cos 3\theta - i\sin 3\theta$$

$$= 2\cos 3\theta + 6\cos\theta$$

$\therefore 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$

$2\cos 3\theta = 8\cos^3\theta - 6\cos\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$